

## The Maximum Principle and Biharmonic Functions\*

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This note concerns the maximum principle which applies to solutions of partial differential equations of elliptic type. This principle asserts that the maximum of a solution occurs on the boundary of a region. Consideration of the ratio of solutions of an elliptic equation shows that the ratio satisfies the same maximum principle. This result is then used to obtain a maximum principle relating to biharmonic functions. These maximum principles give inequalities which biharmonic functions must satisfy. The relations and concepts developed in this note have application in elasticity and in hydrodynamics.

### I. INTRODUCTION

If a function satisfies a partial differential equation of elliptic type (with no undifferentiated term) then the maximum of the function must occur on the boundary of the region. This note concerns applications of this maximum principle.

The first topic treated concerns the ratio of functions which satisfy an elliptic equation. It results that the ratio obeys the same maximum principle. In particular the maximum principle applies to the ratio of harmonic functions.

The last mentioned result is used to obtain maximum principles involving biharmonic functions. Such principles furnish inequalities which biharmonic functions must satisfy.

The concepts and relations developed in this note have applications in elasticity and in hydrodynamics. Some of these applications are treated in another paper [1].

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## II. THE MAXIMUM PRINCIPLE

Of concern are operators of the form

$$L(u) = \sum_1^n \sum_1^n a_{ij} u_{ij} + \sum_1^n b_i u_i. \quad (1)$$

Here  $a_{ij}$  and  $b_i$  are continuous functions of variables  $x_1, \dots, x_n$ . Also  $u$  is a function of these variables and  $u_i$  denotes  $\partial u / \partial x_i$  and  $u_{ij}$  denotes  $\partial^2 u / \partial x_i \partial x_j$ . The second derivatives  $u_{ij}$  are assumed continuous so  $L(u)$  is a continuous function.

The operator  $L$  is said to be elliptic if the  $a_{ij}$  define a symmetric positive definite matrix. If  $L$  is elliptic and  $L(u) \geq 0$  in a region then  $u$  satisfies the strong maximum principle. That is, either  $u$  has no local maxima inside  $R$  or else  $u$  is constant. For short, we say that in such a case  $u$  has no "proper maxima."

Of special concern in this note are singular elliptic operators; the coefficients  $b_i$  are allowed to have infinities. In particular we consider operators of the special form

$$L'(u) = \sum_1^n \sum_1^n a_{ij}(u_{ij} + 2u_i v_j / v) + \sum_1^n b_i u_i. \quad (2)$$

Here  $v$  is a function of class  $C^2$ . Then (2) is seen to be singular at points where  $v$  vanishes. Operators of the type (2) have been considered by Hartman [2]. To limit the degree of the singularity he makes the further hypothesis that

$$\text{grad } v \neq 0 \text{ whenever } v = 0. \quad (3)$$

Then he is able to prove that the maximum principle continues to hold. In other words if  $L'(u) \geq 0$  then  $u$  has no proper maxima.

## III. THE RATIO OF SOLUTIONS

Now of concern are differential equations of the form

$$L(u) = cu. \quad (4)$$

where  $c$  is a continuous function of  $x_1, x_2, \dots, x_n$  and  $L$  is an operator of the form (1). The following result is a formal relationship satisfied by solutions of such second order partial differential equations.

**THEOREM 1.** *Let  $w$  and  $v$  be two functions which satisfy the same partial differentiation of the form (4). Then the ratio  $u = w/v$  satisfies the equation  $L'(u) = 0$  where  $L'$  is of the form (2).*

PROOF: It may be supposed that  $a_{ij} = a_{ji}$ . Then since

$$\frac{\partial u}{\partial x_i} = \frac{vw_i - wv_i}{v^2}$$

and

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{vw_{ij} - wv_{ij} + v_j w_i - w_j v_i}{v^2} - \frac{2u_i v_j}{v}$$

it is seen that

$$L'(u) = L(u) + 2 \sum_1^n \sum_1^n a_{ij} \frac{v_j u_i}{v} = \frac{vL(w) - wL(v)}{v^2}. \quad (2a)$$

The right side vanishes because of (4) and this completes the proof.

**THEOREM 2.** *Let  $w$  and  $v$  be two functions of  $C^2$  which satisfy the same elliptic equation of the form (4). Then the ratio  $u = w/v$  can not have a proper maximum at a point where  $v \neq 0$ . Moreover the latter restriction can be removed if  $u \in C^2$  and  $\text{grad } v \neq 0$  wherever  $v = 0$ .*

PROOF: The first part of Theorem 2 is a consequence of Theorem 1 together with the maximum principle. The second part is seen to follow from the maximum principle for singular equations.

**THEOREM 3.** *Let  $w(x, y, z)$  be a harmonic function. Then  $w/x$  can have no proper maxima.*

PROOF: Theorem 2 is to be applied with  $L$  being the Laplacian and  $v$  being  $x$ . If  $w/x$  has a local maxima at a point of the plane  $x = 0$  then  $w$  must vanish on this plane. But  $w$  is analytic so  $u = w/x$  is analytic and therefore  $u$  is of class  $C^2$ . The conditions of Theorem 2 are verified and the proof is complete.

#### IV. BIHARMONIC FUNCTIONS

The maximum principle for a function  $u$  satisfying the elliptic equation  $Lu = 0$ , has the following corollary: the function  $u$  is positive in a region if it has positive boundary values. This statement suggests a possible form of a maximum principle for biharmonic functions. Thus, let  $w(x, y, z)$  be biharmonic in a region  $R$ , that is  $\Delta\Delta w = 0$  at interior points of  $R$ . Suppose that the Dirichlet type boundary data for  $w$  are positive. Thus  $w > 0$  and  $-\partial w/\partial n > 0$  on the boundary of  $R$  ( $n$  denotes the exterior normal). Then the conjecture is that  $w$  is positive in  $R$ . An equivalent formulation of this conjecture (proposed by Hadamard) is that the biharmonic Green's function is positive.

In a previous paper it was shown that there is a region  $R$  such that the above conjecture is false [3]. Further studies of such regions have been given by Loewner, Szegö, Nehari, and P. Garabedian.

It is the purpose here to reformulate the conjecture so as to obtain valid maximum principles. For the sake of being definite three dimensional space is treated.

**THEOREM 4.** *Let the function  $w(x, y, z)$  be biharmonic in a region  $R$  and let  $w$  together with its first and second derivatives have continuous limits on the boundary  $B$  of  $R$ . Then, if  $(a, b, c)$  is an interior point of  $R$ ,*

$$w_x(a, b, c) \leq \max_B [w_x - (1/2)(x - a)\Delta w]. \quad (5)$$

Here  $w_x$  denotes  $\partial w / \partial x$ .

**PROOF:** Relation (5) is a direct consequence of the maximum principle applied to the harmonic function  $w_x - (1/2)(x - a)\Delta w$ .

**THEOREM 5.** *Under the hypotheses of Theorem 4*

$$|Vw(a, b, c)| \leq \max_B |Vw - (1/2)r\Delta w|. \quad (6)$$

Here  $r$  is the vector with components  $x - a$ ,  $y - b$ , and  $z - c$ .

**PROOF:** Note that  $|Vw - (1/2)r\Delta w|^2$  is a sum of squares of three harmonic functions. But the square of a harmonic function is subharmonic and the sum of subharmonic functions is subharmonic. Thus (6) follows from the maximum principle.

**THEOREM 6.** *Under the hypotheses of Theorem 5*

$$w(a, b, c) \leq \max_B [w - 2r \cdot Vw + (1/2)r^2 \Delta w]. \quad (7)$$

**PROOF:** If  $w$  is an  $n$ -dimensional biharmonic function then it may be verified that

$$(4 - n)w - 2r \cdot Vw + (1/2)r^2 \Delta w$$

is an  $n$ -dimensional harmonic function. Applying the maximum principle in the case  $n = 3$  yields (7). It is to be noted that this method of proof fails in four dimensions.

**THEOREM 7.** *Suppose the hypotheses of Theorem 4 and that  $x \geq 0$  in  $R$ . Then*

$$w_x(a, b, c) \leq \max_B \left[ \frac{a}{x} w_x - \left( \frac{a}{2} - \frac{a^2}{x} \right) \Delta w \right]. \quad (8)$$

PROOF: Theorem 3 applied to the harmonic function  $w_x - (x-a)\Delta w/2$  yields (8).

THEOREM 8. *Suppose the hypotheses of Theorem 4 and that  $w_x = 0$  at the points of  $R$  where  $x = 0$ . Then*

$$w_x(a, b, c) \leq \max_B \left[ \frac{w_x}{x} - \frac{\Delta w}{2} \right] + \max_B \left[ \frac{\Delta w}{2} \right] \quad (9)$$

and if  $a = 0$  the left side is to be interpreted as  $w_{xx}(a, b, c)$ . If relation (9) is an equality then  $w_x/x$  is a constant.

PROOF: The function  $w_x/x$  can be written as a sum of two terms each of which satisfies an elliptic equation. Thus

$$\frac{w_x}{x} = \left[ \frac{w_x}{x} - \frac{\Delta w}{2} \right] + \left[ \frac{\Delta w}{2} \right].$$

The maximum principle applies to the first term in brackets by virtue of Theorem 3. The maximum principle applies to the second term because it is a harmonic function. Then inequality (9) follows by addition. Moreover because of the strong form of the maximum principle, it is clear that (9) is a strict inequality or else both the bracket terms are constant. It then follows that  $w_x/x$  is a constant.

THEOREM 9. *Let  $w(x, y, z)$  and  $w(2a - x, y, z)$  as well, satisfy the hypotheses of Theorem 4. Define*

$$2W(x, y, z) = w(x, y, z) + w(2a - x, y, z).$$

Then

$$w_{xx}(a, b, c) \leq \max_B \left[ \frac{W_x}{x-a} - \frac{\Delta W}{2} \right] + \max_B \left[ \frac{\Delta W}{2} \right]. \quad (10)$$

PROOF: If  $a = 0$  this is a direct consequence of Theorem 8 applied to  $W$ . The general case can then be obtained by use of new translated coordinates with origin at  $(a, 0, 0)$ .

Theorems 4, 5, 7 and 8 all give inequalities for the gradient of a biharmonic function. It is clear that such inequalities can be added to obtain further inequalities.

The Harnack inequality for positive harmonic functions bears a certain resemblance to the maximum principle. In this connection it is of some interest to note that a Harnack inequality for positive biharmonic function has been developed in a previous paper [4].

## V. BIHARMONIC FUNCTIONS IN TWO VARIABLES

The following result concerns biharmonic functions in two variables.

THEOREM 10. *Let  $w(x, y)$  be biharmonic in a region of the plane. Then*

$$q = w_x^2 + w_y^2 - (w_{xx} + w_{yy})w \quad (11)$$

*is subharmonic and so can have no proper maxima*

PROOF: Applying the Laplacian to (11) gives:

$$\Delta(w_x^2 + w_y^2) = 2w_x \Delta w_x + 2w_y \Delta w_y + 2w_{xx}^2 + 2w_{yy}^2 + 4w_{xy}^2$$

$$\Delta(w \Delta w) = (\Delta w)^2 + 2w_x \Delta w_x + 2w_y \Delta w_y + w \Delta^2 w.$$

Subtracting these relations gives

$$\Delta q = (w_{xx} - w_{yy})^2 + 4(w_{xy})^2. \quad (12)$$

Clearly the right side of (12) is nonnegative and the proof is complete.

Consider an inversion transformation in a circle of radius  $b$  about the origin. The biharmonic function  $w(x, y)$  is transformed into a biharmonic function  $w^*(x, y)$  by the well known formula

$$w^*(x, y) = \frac{r^2}{b^2} w\left(\frac{b^2 x}{r^2}, \frac{b^2 y}{r^2}\right), \quad r^2 = x^2 + y^2. \quad (13)$$

THEOREM 11. *The functional  $q$  of Theorem 10 is invariant under an inversion.*

PROOF: It is required to prove that  $q(w)$  at a given point is equal to  $q(w^*)$  at the inverse point. This proof is given by a straightforward substitution of (13) into (11).

*Note added in proof:* Redheffer has called the writer's attention to a recent paper of his [5] which gives a very general analysis of the maximum principle. Theorem VI of that paper and Theorem 2 of this paper overlap. A point of difference is that his work concerns the weak maximum principle while this paper concerns the strong maximum principle.

In a previous note [6] the writer employed a maximum principle to obtain lower bounds for the lowest eigenvalue of Schrödinger's equation. (It will be recalled that the Rayleigh-Ritz method yields upper bounds.) This result is generalized in Redheffer's paper so as to apply to arbitrary elliptic operators. He obtains results such as the following: Suppose that  $w$  satisfies the equation  $Lw = \lambda w$  in a region and vanishes on the

boundary of the region. Then  $\lambda \geq \min (Lv/v)$  where the arbitrary function  $v$  is positive in the region and on the boundary. It is worth noting that this result is a consequence of identity (2a) applied at a point where  $u$  has a maximum.

A letter from Weinstein points out that Theorem 3 can be deduced from the theory of generalized axially symmetric potentials. In particular the paper by Huber [7] is cited.

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